two

the sonar equations

The many phenomena and effects peculiar to underwater sound produce a variety of quantitative effects on the design and operation of sonar equipment. These diverse effects can be conveniently and logically grouped together quantitatively in a small number of units called the *sonar parameters,* which, in turn, are related by the *sonar equations.* These equations are the working relationships that tie together the effects of the medium, the target, and the equipment; they are among the design and prediction tools available to the engineer for underwater sound applications.

The sonar equations were first formulated during World War II (1) as the logical basis for calculations of the maximum range of sonar equipments. In recent years, they have seen increasing use in the optimum design of sonars for new applications. Essentially the same relationships are employed in radar (2), though with linear instead of logarithmic units and with slightly different definitions of the parameters.

The essentially simple sonar equations serve two important practical functions. One is the *prediction of performance* of sonar equipments of known or existing design. In this application the design characteristics of the sonar set are known or assumed, and what is desired is an estimate of performance in some meaningful terms such as detection probability or search rate. This is achieved in the sonar equations by a prediction of range through the parameter transmission loss. The equations are solved for transmission loss, which is then converted to range through some assumption concerning the propagation characteristics of the medium.

The other general application of the equations is in *sonar design,* where a preestablished range is required for the operation of the equipment being designed. In this case the equation is solved for the particular troublesome parameter whose practical realization is likely to cause difficulty. An example would be the directivity required, along with other probable values of sonar parameters, to yield a desired range of detection in a detection sonar or the range of actuation by a passing ship of an acoustic mine mechanism. After the directivity needed to obtain the desired range has been found, the design continues through the "trade-offs" between directivity index and other parameters. The design is finally completed through several computations using the equations and the design engineer's intuition and experience.

2.1 Basic Considerations

The equations are founded on a basic equality between the desired and undesired portions of the received signal at the instant when some function of the sonar set is just performed. This function may be detection of an underwater target, or it may be the homing of an acoustic torpedo at the instant when it just begins to acquire its target. These functions all involve the reception of acoustic energy occurring in a natural acoustic background. Of the total acoustic field at the receiver, a portion may be said to be *desired* and is called the *signal.* The remainder of the acoustic field is *undesired* and may be called the *background.* In sonar the background is either *noise,* the essentially steady-state portion not due to one's own echo ranging, or *reverberation,* the slowly decaying portion of the background representing the return of one's own acoustic output by scatterers in the sea. The design engineer's objective is to find means for *increasing* the overall response of the sonar system to the signal and for *decreasing* the response of the system to the background—in other words, to *increase* the signal-to-background ratio.

Let us imagine a sonar system serving a practical purpose such as *detection, classification* (determining the nature of a target), *torpedo homing, communication,* or *fish finding.* For each of these purposes there will be a certain signal-to-background ratio that will depend on the functions being performed and on the performance level that is desired in terms of percentages of successes and "false alarms," such as an apparent detection of a target when no target is present. If the signal is imagined to be slowly increasing in a constant background, the desired purpose will be accomplished when the *signal level equals the level of the background which just masks it.* That is to say, when the sonar's purpose is *just* accomplished,

Signal level = background masking level

The term "masking" implies that not all the background interferes with the signal, but only a portion of it—usually that portion lying in the frequency band of the signal. The word "masking" is borrowed from the theory of audition, where it refers to that part of a broadband noise background that masks out a pure tone or a narrow-band signal presented to a human listener.

We should note that the equality just stated will exist at only *one* instant of time when a target approaches, or recedes from, a sonar receiver. At short ranges, its signal level will exceed the background masking level; at long ranges, the reverse will occur. But the instant of equality is the moment of greatest interest to the sonar engineer or designer, for it is at this instant that the sonar system *just* performs its assigned function. It is this instant to which the engineer or designer will often focus attention in a sonar calculation.

2.2 The Active and Passive Equations

The next step is to expand the basic equality in terms of the *sonar parameters* determined by the *equipment*, the *medium*, and the *target*. We will denote these parameters by two-letter symbols in order to avoid Greek and subscripted symbols as much as possible in the writing of the equations. These parameters are levels in units of decibels relative to the standard reference intensity of a 1-μPa plane wave. They are as follows:

Parameters Determined by the *Equipment*
  Projector Source Level: SL
  Self-Noise Level: NL
  Receiving Directivity Index: DI
  Detection Threshold: DT
Parameters Determined by the *Medium*
  Transmission Loss: TL
  Reverberation Level: RL
  Ambient-Noise Level: NL
Parameters Determined by the *Target*
  Target Strength: TS
  Target Source Level: SL

Two pairs of the parameters are given the same symbol because they are essentially identical. It should be mentioned in passing that this set of parameters is not unique. Others, which could be employed equally well, might be more fundamental or might differ by a constant. For example, sound velocity could be adopted as a parameter, and TS could be replaced by the parameter "backscattering cross section" expressed in decibels, as is done in radar. The chosen parameters are therefore arbitrary; those employed here are the ones conventionally used in underwater sound.* It should also be noted that they may all be expanded in terms of fundamental quantities like frequency, ship speed, and bearing—a subject that will be of dominant importance in the descriptions of the parameters that will follow. The units of the parameters are decibels, and they are added together in forming the sonar equations.

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* There is, however, no conventional symbolism for many of the parameters.

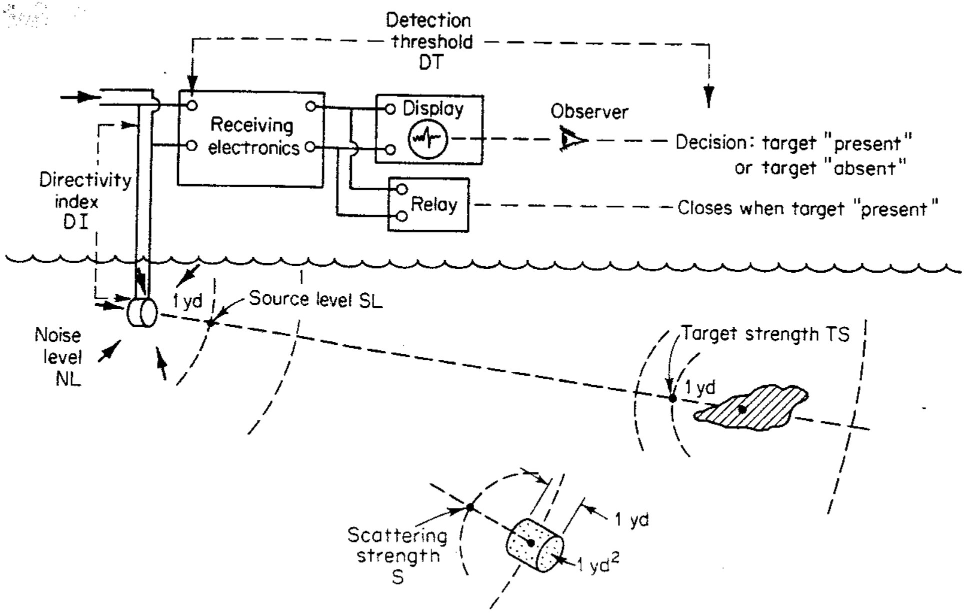

fig. 2.1 *Diagrammatic view of echo ranging, illustrating the sonar parameters.*

The meaning of these quantities can best be illustrated through some simple considerations for an active (echo-ranging) sonar (Fig. 2.1). A sound source acting also as a receiver (a *transducer*) produces by some means a *source level* of SL decibels at a unit distance (1 yd) on its axis. When the radiated sound reaches the target (if the axis of the sound source points toward the target), its level will be reduced by the *transmission loss,* and becomes SL − TL. On reflection or scattering by the target of target strength TS, the reflected or backscattered level will be SL − TL + TS at a distance of 1 yd from the acoustic center of the target in the direction back toward the source. In traveling back toward the source, this level is again attenuated by the *transmission loss* and becomes SL − 2TL + TS. This is the echo level at the transducer. Turning now to the background and assuming it to be isotropic noise rather than reverberation, we find that the *background level* is simply NL. This level is reduced by the *directivity index* of the transducer acting as a receiver or *hydrophone* so that at the terminals of the transducer the relative noise power is NL − DI. Since the axis of the transducer is pointing in the direction from which the echo is coming, the relative echo power is unaffected by the transducer directivity. At the transducer terminals, therefore, the echo-to-noise ratio is

$$\mathrm{SL} - 2\mathrm{TL} + \mathrm{TS} - (\mathrm{NL} - \mathrm{DI})$$

Let us now assume that the function that this sonar is called upon to perform is *detection,* that is, that its principal purpose is to give an indication of some sort on its *display* whenever an echoing target is present. When the input signal-to-noise ratio is above a certain detection threshold fulfilling certain probability criteria, a decision will be made by a human observer that a target is *present**; when the input signal-to-noise ratio is less than the detection threshold, the decision will be made that the target is *absent*. When the target is *just* being detected, the signal-to-noise ratio *equals* the detection threshold, and we have

$$\mathrm{SL} - 2\mathrm{TL} + \mathrm{TS} - (\mathrm{NL} - \mathrm{DI}) = \mathrm{DT}$$

We have here the active-sonar equation as an equality in terms of the *detection threshold,* called in audition and in much of the older underwater sound literature *recognition differential.* In terms of the basic equality described above, we could equally well consider that only that part of the noise power lying above the detection threshold masks the echo, and we would then have

$$\mathrm{SL} - 2\mathrm{TL} + \mathrm{TS} = \mathrm{NL} - \mathrm{DI} + \mathrm{DT}$$

a more convenient arrangement of the parameters, since the echo level occurs on the left-hand side and the noise-masking background level occurs on the right.

This is the active-sonar equation for the *monostatic case* in which the source and receiving hydrophone are coincident and in which the acoustic return of the target is back toward the source. In some sonars, a separated source and receiver are employed and the arrangement is said to be *bistatic;* in this case, the two transmission losses to and from the target are not, in general, the same. Also in some modern sonars, it is not possible to distinguish between DI and DT, and it becomes appropriate to refer to DI − DT as the increase in signal-to-background ratio produced by the entire receiving system of transducer, electronics, display, and observer (if one is used).

A modification is required when the background is reverberation instead of noise. In this case, the parameter DI, defined in terms of an isotropic background, is inappropriate, inasmuch as reverberation is by no means isotropic. For a reverberation background we will replace the terms NL − DI by an *equivalent plane-wave reverberation level* RL observed at the hydrophone terminals. The active-sonar equation then becomes

$$\mathrm{SL} - 2\mathrm{TL} + \mathrm{TS} = \mathrm{RL} + \mathrm{DT}$$

where the parameter DT for reverberation has in general a different value than DT for noise.

In the passive case, the target itself produces the signal by which it is detected, and the parameter source level now refers to the level of the radiated noise of the target at the unit distance of 1 yd. Also, the parameter target strength becomes irrelevant, and one-way instead of two-way transmission is involved. With these changes, the *passive-sonar equation* becomes

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* If the human observer is replaced by a relay at the output of the detector, then the detection threshold is the input signal-to-background ratio at the transducer terminals which *just* closes the relay to indicate "target present."

$$\mathrm{SL} - \mathrm{TL} = \mathrm{NL} - \mathrm{DI} + \mathrm{DT}$$

Table 2.1 is a list of parameters, reference locations, and short definitions in the form of ratios. More complete definitions of the parameters will be given near the beginnings of the chapters dealing with the parameters.

2.3 Names for Various Combinations of Parameters

In practical work it is convenient to have separate names for different combinations of the terms in the equations. Methods exist for measuring some of these on shipboard sonars as a check on system operation. Table 2.2 is a listing of these names and the combination of terms that each represents. Of these, *the figure of merit* is the most useful, because it combines together the various equipment and target parameters so as to yield a quantity significant for the performance of the sonar. Since it equals the transmission loss at the instant when the sonar equation is satisfied, the figure of merit gives an immediate indication of the range at which a sonar can detect its target, or more generally, perform its function. However, when the background is reverberation instead of noise, the figure of merit is not constant, but varies with range and so fails to be a useful indicator of sonar performance.

table 2.1 The Sonar Parameters, Their Definitions, and Reference Locations

Parameter symbol		Reference	Definition
Source level	SL	1 yd from source on its acoustic axis	$10 \log \dfrac{\text{intensity of source}}{\text{reference intensity}^*}$
Transmission loss	TL	1 yd from source and at target or receiver	$10 \log \dfrac{\text{signal intensity at 1 yd}}{\text{signal intensity at target or receiver}}$
Target strength	TS	1 yd from acoustic center of target	$10 \log \dfrac{\text{echo intensity at 1 yd from target}}{\text{incident intensity}}$
Noise level	NL	At hydrophone location	$10 \log \dfrac{\text{noise intensity}}{\text{reference intensity}^*}$
Receiving directivity index	DI	At hydrophone terminals	$10 \log \dfrac{\text{noise power generated by an equivalent nondirectional hydrophone}}{\text{noise power generated by actual hydrophone}}$
Reverberation level	RL	At hydrophone terminals	$10 \log \dfrac{\text{reverberation power at hydrophone terminals}}{\text{power generated by signal of reference intensity}^*}$
Detection threshold	DT	At hydrophone terminals	$10 \log \dfrac{\text{signal power to just perform a certain function}}{\text{noise power at hydrophone terminals}}$

* The reference intensity is that of a plane wave of rms pressure 1 μPa.

table 2.2 Terminology of Various Combinations of the Sonar Parameters

Name	Parameters	Remarks
Echo level	SL − 2TL + TS	The intensity of the echo as measured in the water at the hydrophone
Noise-masking level	NL − DI + DT	Another name for these two combinations is *minimum detectable echo level*
Reverberation-masking level	RL + DT	
Echo excess	SL − 2TL + TS − (NL − DI + DT)	Detection just occurs, under the probability conditions implied in the term DT, when the echo excess is zero
Performance figure	SL − (NL − DI)	Difference between the source level and the noise level measured at the hydrophone terminals
Figure of merit	SL − (NL − DI + DT)	Equals the maximum allowable one-way transmission loss in passive sonars, or the maximum allowable two-way loss for TS = 0 dB in active sonars

2.4 The Parameters in Metric Units

Some of the parameters of Table 2.1 have 1 yard as their reference distance. These are SL, TL, TS, and (as a determining quantity for RL) the scattering strength S. If, instead, 1 meter is taken as the reference distance, and it is desired to use metric units in a calculation involving the sonar equations, these quantities should be *decreased* by the amount 20 log (1 meter/1 yard) = 0.78 dB. In addition, the attenuation coefficient, commonly expressed in English units in decibels per kiloyard, must be multiplied by 1.094 to convert it to decibels per kilometer. No other sonar quantities are affected by a choice of units; nor are the quantities echo level, noise-masking level, and echo excess (defined in Table 2.2). For finding sonar ranges in metric units, it is often more convenient to find the range first in kiloyards, and then to divide by the factor 1.094 to obtain the range in kilometers.

2.5 Echo, Noise, and Reverberation Level as Functions of Range

The sonar equations just written are no more than a statement of an equality between the desired portion of the acoustic field called the *signal*—either an echo or a noise from a target—and an undesired portion, called the *background* of noise or reverberation. This equality, in general, will hold at only one range; at other ranges, one or the other will be the greater, and the equality will no longer exist.

This is illustrated in Fig. 2.2, where curves of echo level, noise-masking level, and reverberation-masking level are shown as a function of range. Both the echo and reverberation fall off with range, whereas the noise remains

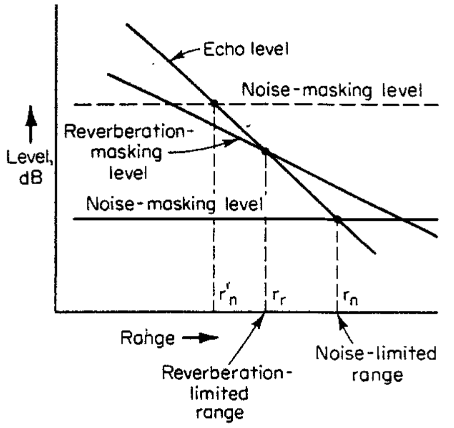

fig. 2.2 Echo, noise, and reverberation as functions of range.

constant. The echo-level curve will generally fall off more rapidly with range than the reverberation-masking level curve and will intersect it at the *reverberation-limited range* r_r given by the sonar equation for reverberation. The curve of echo level will also intersect the noise-masking level at the range of the sonar equation for noise r_n. If the reverberation is high, the former will be less than the latter, and the range will be said to be *reverberation-limited.* If for any reason the noise-masking level rises to the level shown by the dashed line in the figure, the echoes will then die away into a background of noise rather than reverberation. The new noise-limited range r'_n will then be less than the reverberation-limited range r_n, and the range will become *noise-limited.* Both ranges are given by the appropriate form of the sonar equation.

A knowledge of whether a sonar will be noise- or reverberation-limited is necessary for both the sonar predictor and the sonar designer. In general, the curves for echo and reverberation will not be straight lines because of complications in propagation and in the distribution of reverberation-producing scatterers. For a new sonar system, such curves should always be drawn from the best information available for the conditions most likely to be encountered in order to demonstrate visually to the design engineer the behavior of the signals and background with range.

For passive sonars, a convenient graphical way of solving the sonar equations is called SORAP, denoting "sonar overlay range prediction." It consists of two plots that are laid one upon the other (Fig. 2.3). The overlay (solid lines) is a plot of SL versus frequency for a particular passive target or class of targets; the underlay (dashed lines) is a plot of the sum of the parameters TL + NL − DI + DT for a particular passive sonar and for a number of different ranges. The range and frequency at which the target can be detected can be readily read off by inspection. In Fig. 2.3, where the two plots are superposed, the target will be detected first at a range of 10 miles by means of the line component at frequency f_1. However, if the criterion is adopted that *three* spectral lines must appear on the display, whatever it may be, before a detec-

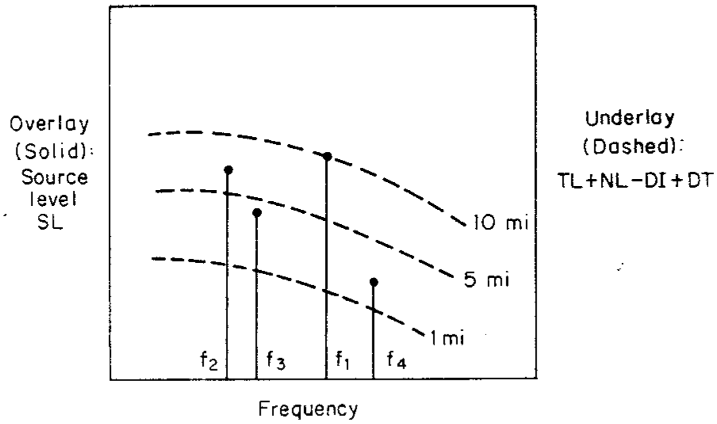

fig. 2.3 *SORAP: a graphical way to solve the passive sonar equation.*

tion is called, the range would be reduced to 4 miles, and the display would show the lines at frequencies f_1, f_2, and f_3. The method is particularly useful for separating the target parameter SL from the equipment parameters and medium parameters at the location where it is used, while at the same time accommodating a wide range of frequencies. Thus, targets can be compared for the same equipment and locations, or, alternatively, locations can be compared for the same target, and so forth.

2.6 Transient Form of the Sonar Equations

The equations thus far have been written in terms of *intensity*, or the average acoustic power per unit area of the sound emitted by the source or received from the target. The word "average" implies a time interval over which the average is to be taken. This time interval causes uncertain results whenever short transient sources exist or, generally, whenever severe distortion is introduced by propagation in the medium or by scattering from the target.

A more general approach is to write the equations in terms of *energy flux density*, defined as the acoustic energy per unit area of wavefront (see Sec. 1.5). If a plane acoustic wave has a time-varying pressure $p(t)$, then the energy flux density of the wave is

$$E = \frac{1}{\rho c}\int_0^\infty p^2(t)\,dt$$

If the units of pressure are dynes per square centimeter and the acoustic impedance of the medium is in cgs units (for water, $\rho c \approx 1.5 \times 10^5$), then E will be expressed in ergs per square centimeter. The intensity is the mean-square pressure of the wave divided by ρc and averaged over an integral of time T, or

$$I = \frac{1}{T}\int_0^T \frac{p^2(t)}{\rho c}\,dt$$

so that over the time interval T,

$$I = \frac{E}{T}$$

The quantity T is accordingly the time interval over which the energy flux density of an acoustic wave is to be averaged to form the intensity. For long-pulse active sonars, this time interval is the duration of the emitted pulse and is very nearly equal to the duration of the echo. For short transient sonars, however, the interval T is often ambiguous, and the duration of the echo is vastly different from the duration of the transient emitted from the source. Under these conditions, however, it can be shown (3) that the intensity form of the sonar equations can be used, provided that the source level is defined as

$$\text{SL} = 10 \log E - 10 \log \tau_e$$

where E is the energy flux density of the source at 1 yd and is measured in units of the energy flux density of a 1-μPa plane wave taken over an interval of 1 second, and τ_e is the duration of the echo in seconds for an active sonar. For explosives, E is established by measurements for a given charge weight, depth, and type of explosive (Sec. 4.4). For pulsed sonars emitting a flat-topped pulse of constant source level SL′ over a time interval τ_0, then, since the energy density of a pulse is the product of the average intensity times its duration,

$$10 \log E = \text{SL}' + 10 \log \tau_0$$

By combining the last two equations, the effective source level SL for use in the sonar equations is therefore

$$\text{SL} = \text{SL}' + 10 \log \frac{\tau_0}{\tau_e}$$

Here τ_0 is the duration of the emitted pulse of source level SL′, and τ_e is the echo duration. For long-pulse sonars, $\tau_0 = \tau_e$ and SL = SL′. For short-pulse sonars, $\tau_e > \tau_0$, and the effective source level SL is less than SL′ by the amount $10 \log (\tau_0/\tau_e)$. The effect of time stretching on source level may be visualized as shown in Fig. 2.4. A short pulse of duration τ_0 and source level SL′ is replaced in a sonar calculation by an effective or equivalent pulse of longer duration τ_e and lower source level SL. The two source levels are related so as to keep the energy flux-density source levels the same, namely:

$$\text{SL} + 10 \log \tau_e = \text{SL}' + 10 \log \tau_0$$

or

$$\text{SL} = \text{SL}' + 10 \log \frac{\tau_0}{\tau_e}$$

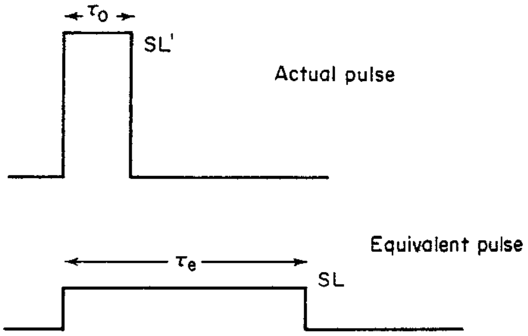

***fig. 2.4** Equivalent source level in short-pulse sonars.*

In effect, the pulse emitted by the source is stretched out in time and thereby reduced in level by the multipath effects of propagation and by the processes of target reflection. The appropriate values of other sonar parameters in the equations, such as TS and TL, are those applying for long-pulse or CW conditions, in which the effects of multipaths in the medium and on the target are added up and accounted for.

For active short-pulse sonars, the echo duration τ_e is, accordingly, a parameter in its own right. Figure 2.5 illustrates a pulse as a short exponential transient at the source, as a distorted pulse at the target, and as an echo received back in the vicinity of the source. An exponential pulse, similar in form to the shock wave from an explosion of about 1 lb of TNT and having an initial duration of 0.1 ms, becomes distorted into an echo 1,000 times as long. Two actual examples of explosive echoes are shown in Fig. 2.6.

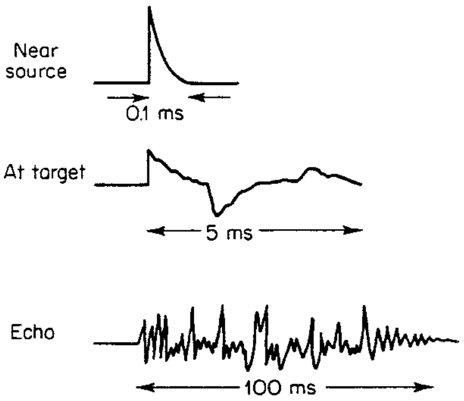

***fig. 2.5** Diagrams of the pressure of an explosive pulse near the source, on arrival at an extended target, and as an echo back in the vicinity of the source.*

The echo duration can be conceived as consisting of three components: τ_0, the duration of the emitted pulse measured near the source; τ_m, the additional duration imposed by the two-way propagation in the sea; and τ_t, the additional duration imposed by the extension in range of the target. In this view, the echo duration is the sum of the three components, or

$$\tau_e = \tau_0 + \tau_t + \tau_m$$

Typical examples of the magnitude of these three components of the echo duration under different conditions are given in Table 2.3. Thus, with these

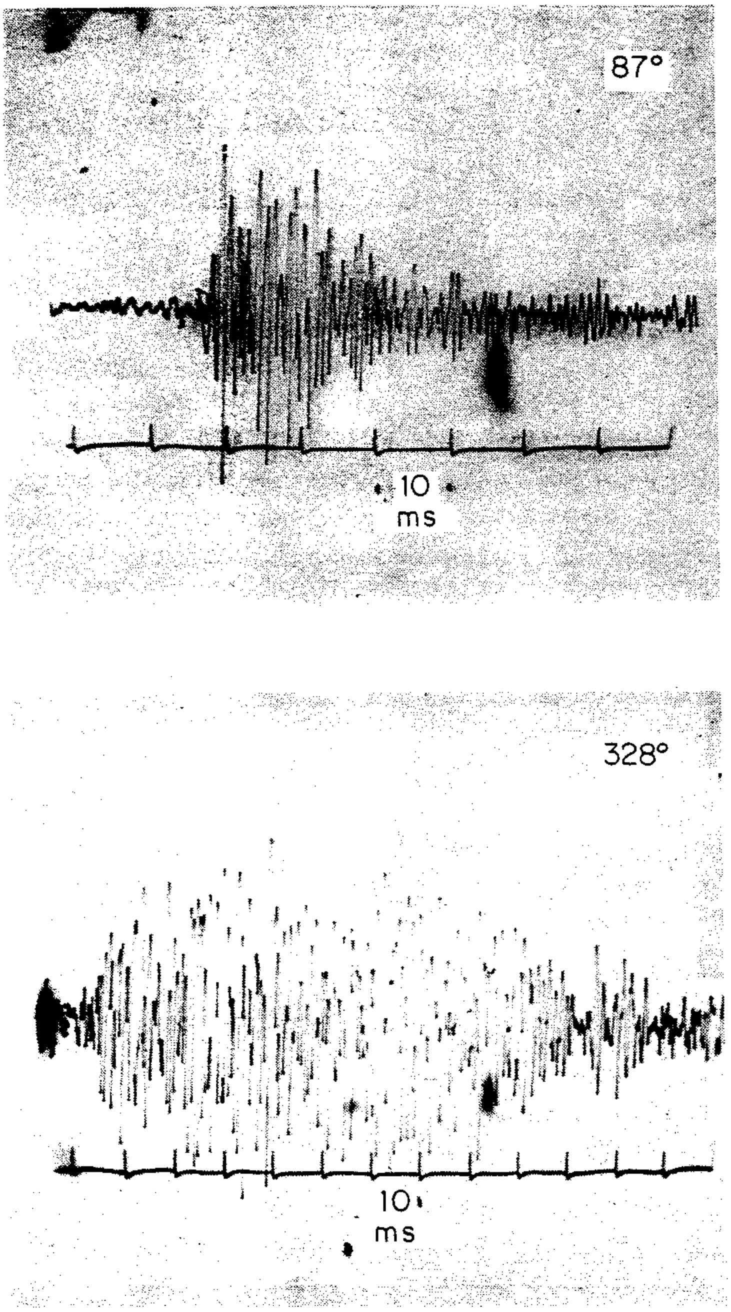

fig. 2.6 *Oscilloscope photographs of explosive echoes from a submarine at aspect angles 87° (near-beam aspect) and 328° (near-bow aspect). The time ticks are 10 ms apart.*

table 2.3 Components of Echo Duration

Component	Typical values, ms
Duration of the emitted pulse at short ranges	Explosives: 0.1 Sonar: 100
Duration produced by multiple paths	Deep water: 1 Shallow water: 100
Duration produced by a submarine target	Beam aspect: 10 Bow-stern aspect: 100

values, the time duration of an explosive echo from a bow-stern aspect submarine in shallow water would be 0.1 + 100 + 100 = 200.1 ms.

2.7 Statement of the Equations

A condensed statement of the equations is as follows.

Active sonars (monostatic):

Noise background

$$SL - 2TL + TS = NL - DI + DT$$

Reverberation background

$$SL - 2TL + TS = RL + DT_R$$

Passive sonars:

$$SL - TL = NL - DI + DT_N$$

where DT has been subscripted to make it clear that this parameter is quantitatively different for noise and for reverberation.

2.8 Limitations of the Sonar Equations

The sonar equations written in terms of intensities are not always complete for some types of sonars. We have already seen how short-pulse sonars require the addition of another term, the echo duration, to account for the time-stretching produced by multipath propagation. Another such addition is a *correlation loss* in correlation sonars to account for the decorrelation of the signal relative to a stored replica; such decorrelation occurs on bottom reflection and scattering in bottom-bounce sonars. Other terms may be conceivably required for other sophisticated sonars whose operation does not depend on intensity alone.

A limitation of another kind is produced by the nature of the medium in which sonars operate. The sea is a moving medium containing inhomogeneities of various kinds, together with irregular boundaries, one of which is in motion. Multipath propagation is the rule. As a result, many of the sonar parameters fluctuate irregularly with time, while others change because of unknown changes in the equipment and the platform on which it is mounted. Because of these fluctuations, a "solution" of the sonar equations is no more than a best-guess time average of what is to be expected in a basically stochastic problem.

Precise calculations, to tenths of decibels, are futile; a predicted sonar range is an average quantity about which the observed values of range are likely to congregate. We can hope that as our knowledge of underwater sound and its fluctuations improves, the accuracy of the predictions of the sonar equations can be expected to increase.